The vector in Span$\{\mathbf{u}_1\}$ is

$$\frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 = \frac{14}{7}\mathbf{u}_1 = 2\mathbf{u}_1 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}$$

Since $\mathbf{x} = \dfrac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$, the vector

$$\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix}$$

is in Span$\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = -12 + 12 + 0 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. The orthogonal projection of $\mathbf{y}$ onto Span$\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 = \frac{30}{25} \mathbf{u}_1 - \frac{15}{25} \mathbf{u}_2 = \frac{6}{5} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} - \frac{3}{5} \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

8. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = -1 + 3 - 2 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. By the Orthogonal Decomposition Theorem,

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 = 2\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2 = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix}, \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$$

and $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

12. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. The Best Approximation Theorem says that $\hat{\mathbf{y}}$, which is the orthogonal projection of y onto $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, is the closest point to $\mathbf{y}$ in W. This vector is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 = 3\mathbf{v}_1 + 1\mathbf{v}_2 = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$$

19. By the Orthogonal Decomposition Theorem, $\mathbf{u}_3$ is the sum of a vector in $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and a vector $\mathbf{v}$ orthogonal to W. This exercise asks for the vector $\mathbf{v}$:

$$\mathbf{v} = \mathbf{u}_3 - \text{proj}_W \mathbf{u}_3 = \mathbf{u}_3 - \left(-\frac{1}{3}\mathbf{u}_1 + \frac{1}{15}\mathbf{u}_2 \right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ -2/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/5 \\ 1/5 \end{bmatrix}$$

Any multiple of the vector $\mathbf{v}$ will also be in $W^\perp$.

20. By the Orthogonal Decomposition Theorem, $\mathbf{u}_4$ is the sum of a vector in $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and a vector $\mathbf{v}$ orthogonal to W. This exercise asks for the vector $\mathbf{v}$:

$$\mathbf{v} = \mathbf{u}_4 - \text{proj}_W \mathbf{u}_4 = \mathbf{u}_4 - \left(\frac{1}{6}\mathbf{u}_1 - \frac{1}{30}\mathbf{u}_2 \right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/5 \\ -2/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 4/5 \\ 2/5 \end{bmatrix}$$

Any multiple of the vector $\mathbf{v}$ will also be in $W^\perp$.